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Remarks on Unified Field Structures, Spin Structures and Canonical Quantization†

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Abstract

Unified field structures are defined and reviewed. Under certain conditions these are shown to be dynamical systems. And quantizable dynamical systems are shown to be unified field structures with invariant Riemannian metric. Spin structure is reviewed and manifolds M^{8k+4} with spin structure are shown to be symplectic.

1. *Introduction*

A *dynamical system* (D.S.) is pair (E^{2n+1},Ω), smooth ($=C^\infty$) manifold E, $\dim E = 2n+1$, and 2-form Ω of rank $2n$ (i.e. $(\Omega)^n \equiv \underbrace{\Omega \wedge \cdots \wedge \Omega}_{n} \neq 0$). If $d\Omega = 0$, then (E^{2n+1},Ω) is a D.S. with integral invariant (D.S.I.). E^{2n+1} is an *almost contact manifold* if there is a 1-form ω and a 2-form π such that $\omega \wedge (\pi)^n \neq 0$ at every point of E. Pair (E^{2n+1},ω) is a *contact manifold* if $\omega \wedge (d\omega)^n \neq 0$ at every point of E. In either case, there is uniquely associated a canonical vector field $C \in v(E)$ of Cartan-Reeb characterized by $i(C)\omega = 1$ and $i(C)\pi = 0$ (resp. $i(C)d\omega = 0$). In the latter case, Cartan's identity, $\mathscr{L}(X)\omega = i(X)d\omega + di(X)\omega$, for $X = C$ gives $\mathscr{L}(C)\omega = 0$—where $\mathscr{L}$ is the Lie derivative, i is the inner product and d is the exterior derivative; this states that C is an infinitesimal automorphism of the contact structure. If (E,ω) is a contact manifold, then $\Omega = d\omega$ has rank $2n$ and $d\Omega = 0$, so (E,Ω) is a D.S.I.

In two recent studies (Hurt, 1968, 1970b) I defined a quantizable D.S. (Q.D.S.) as the principal toral bundle $\xi : T \to E \xrightarrow{p} M$ over (symplectic) manifold M, where E carries 'dynamic' contact structure ω; that is, ω defines a connection on E. C is non-zero, vertical (i.e. tangent to the fiber at each point), and generates on E an effective, transitive action of the torus T as group of diffeomorphisms. The orbit space $M = E/C$ is symplectic (Hausdorff manifold), since projection of $\Omega = d\omega$ on M is closed of maximum rank (i.e. $d\Omega = 0$, $(\Omega)^n \neq 0$).

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Structures of type ξ have a considerable history in twentieth-century physics. Kaluza (1921) and Klein (1926) introduced a five-dimensional 'stationary' Riemannian manifold E—i.e. a Riemannian manifold admitting a (unit) vector field C—so 1-parameter group $\phi_t = \exp(tC)$, with respect to which the five-dimensional Riemannian metric is invariant. The orbit space $M = E/C$ of this group is a four-dimensional manifold which is taken as space-time, the projective space for general relativity. Modifications were made by Einstein & Mayer (1931–2). Veblen & Hofmann (1930, 1933), van Dantzig (1932), Schouten & van Dantzig (1933), Schouten (1935) and with Haantjes (1936), Haantjes (1937), Thomas (1926, 1934), Whitehead (1936), following Weyl (1921) and Eddington (cf. Eisenhart, Veblen and T. Y. Thomas in the 1920s), and Cartan (1924) introduced projective relativity theories of the form ξ where E is an $n+1$-dimensional manifold with affine connection admitting a 'concurrent' vector field C [i.e. locally the line element of E has the form $ds^2 = (dx^{n+1})^2 + G_{ab}(x^a)\,dx^a\,dx^b$ where, $a, b = 1, \ldots, n$, and $G_{ab}(x^a)$ is a Riemannian metric on some n-dimensional subspace] with respect to which the affine connection is invariant; and M, the n-dimensional manifold with projective connection is the space of orbits of C. Vranceanu (1936) and Yano (1937, 1938a) identified space-time with a non-holonomic hypersurface ('distribution') in E. Cf. also Pauli (1933), Einstein-Bergmann (1938), Jordan (1947*), Bergmann (1948), Lichnerowicz-Thiry (1947*), Thiry (1951), Yano-Ohgane (1952, 1955) and references therein; surveys of the status of these theories—Jordan-Thiry and Einstein-Schrödinger—are in texts of Pauli, Bergmann and Lichnerowicz (1955), where those references marked with an asterisk are given.

Clearly then, ξ is the basic structure of unified field theory, called fibred spaces of certain types and studied by Mutō (1952), Yano & Davies (1959) and Yano & Ishihara (1966, 1967a, b, c), cf. also the general formalism of Souriau (1958, 1964) to treat five-dimensional theories.

2. *Unified Field Structures*

The structure of unified field theory is given by data (E, M, p, C, ω; ∇ or g); namely, a pair of smooth manifolds E^{n+1}, M^n under a smooth, onto map p of maximum rank ($=\dim M$), thus a fiber bundle by Ehresmann; C a non-zero vector field on E tangent to each fiber everywhere (=vertical); 1-form ω on E such that $\omega(C) = 1$ and $\mathscr{L}(C)\,\omega = 0$; and finally E has either an (torsionless) affine connection ∇ invariant by $\mathscr{L}(C)$ or an invariant, positive definite Riemannian metric g—i.e. $\mathscr{L}(C)g = 0$ and $g(C, C) = 1$, in which case E has invariant, torsionless affine connection, namely the Riemannian connection given by g, and 1-form ω is taken to be $\omega(X) = g(X, C)$, $X \in v(E)$ [so $\omega(C) = 1$ and $\mathscr{L}(C)\,\omega = 0$]. Let us abbreviate this by *unified field structure* (of a certain type: invariant affine connection or invariant Riemannian connection).

In either case, the 2-form $\Omega = d\omega$ is closed (as is its projection on M). [If locally $\omega = p_i\,dx^i$ (Einstein summation convention), p_i local components of C, then $p^i p_i = 1$, where $p_i = g_{ij} p^j$, denoted 'E_i' by Yano (1952, etc.), is just the statement $\omega(X) = g(X, C)$, $\omega(C) = 1$; and $\Omega = d\omega = \frac{1}{2}\phi_{ij} dx^i \wedge dx^j$ where $\phi_{ij} = \partial_i p_j - \partial_j p_i$, $\partial_i = \partial/\partial x^i$.] Thus we have

Proposition 1. If ξ is a unified field structure and if $\Omega = d\omega$ is of maximum rank n, then (E^{n+1}, Ω) is a D.S.I.

Sasaki (1960) showed that (E, ω, π) is an almost contact manifold iff there is tensor field ϕ of type (1,1) over E, with C and ω as above, such that $\omega(C) = 1$ and $\phi \circ \phi = -\mathbf{1} + \omega . C$ (also, $\phi(C) = 0$, $\omega \circ \phi = 0$). Every almost contact manifold admits a positive definite Riemannian metric g such that $g(X, C) = \omega(X)$ and $g(\phi X, \phi Y) = g(X, Y) - \omega(X)\omega(Y)$ for $X, Y \in v(E)$. Data (ϕ, C, ω, g) is then called almost contact metric (Riemannian) structure. Defining Ω by $\Omega(X, Y) = g(X, \phi Y)$, then Ω is 2-form of rank $2n$. By Sasaki (1960) and Hatakeyama (1962) we have

Proposition 2. If (E, Ω) is a D.S., then E admits an almost contact metric structure (ϕ, C, ω, g) such that $\Omega(X, Y) = g(X, \phi Y)$.

Corollary 3. If (E, ω) is a contact manifold, then there is a ϕ and g on E such that (ϕ, C, ω, g) is an almost contact metric structure such that $\omega(C) = 1$, $d\omega(C, X) = 0$ and $d\omega(X, Y) = g(X, \phi Y)$.

Remarks. (1) If locally a tensor field ϕ is $\phi_j{}^i$, then $\phi_{ij} = g_{ih}\phi_j{}^h$ is a skew-symmetric tensor of rank $2n$; $\Omega = \frac{1}{2}\phi_{ij} dx^i \wedge dx^j$. (2) If $m^a = m^a(x^i)$, $a, b = 1, \ldots, n$, are local coordinates of M, let $p_i{}^a = \partial_i m^a$ and note that $p^i p_i{}^a = 0$, etc. And G_{ab} of Section 1 is given by $G^{ab} = p_i{}^a p_j{}^b g^{ij}$. (3) Equations for geodesics (Maxwell's equations, trajectory of test particle, variational principle and field equations) are given in works of Yano and co-workers (1952, 1967); cf. Lichnerowicz (1955).

An almost contact, metric structure (ϕ, C, ω, g) such that C is a Killing vector with respect to g, i.e. $\mathscr{L}(C)g = 0$, is called a K-contact structure. By the result of Hatakeyama (1963) we have

Proposition 4. If ξ is a Q.D.S. (i.e. E is a regular or 'dynamic' contact manifold), then (ϕ, C, ω, g) is a K-contact structure.

Thus from Cor. 3, Prop. 4 and the remarks in Section 1, we have

Proposition 5. If ξ is a Q.D.S., then ξ is a unified field structure with invariant Riemannian metric.

As a partial converse, Yano & Ishihara (1967b) have shown

Proposition 6. Unified field structure ξ with invariant Riemannian metric g on E is a K-contact manifold for data $[\phi = (1/l)H, C, \omega, g]$, where H is a second fundamental tensor in E [i.e. $HX = -\nabla_{X^h} C$, X^h denoting the horizontal part of $X \in v(E)$] and l is a non-zero constant, iff the sectional curvature with respect to any section containing C is a constant ($=l^2$) everywhere in E iff pair $\{pg, p[(1/l)H]\}$ is an almost Kählerian structure.

Examples. Yano & Ishihara (1967b) have shown that if ξ is unified field structure with invariant Riemannian metric and non-zero constant curvature k, then M is even dimensional (E odd dimensional), k is positive, and induced structure $\{pg, p[(1/l)H]\}$ is a Kählerian structure of positive constant holomorphic sectional curvature k_M; $k = n(n+1)l^2$, $k_M = n(n+2)l^2$. So for odd dimensional sphere S^{2n+1} with natural Riemannian metric g of positive constant curvature 1, if $p: S^{2n+1} \to M^{2n}$ is a unified field structure with invariant Riemannian metric g, then $M^{2n} = P^n(\mathbf{C})$, p is the natural projection, and pg gives the natural Kählerian structure of positive constant holomorphic sectional curvature. $\xi: T \to S^{2n+1} \to P^n(\mathbf{C})$ (Hopf fibering) is a Q.D.S., namely $n+1$ independent harmonic oscillators with equal periods, cf. Hurt (1970a).

3. *Spin Structures*

The basic structure to general relativity (cf. Lichnerowicz *et al.*, 1964) is an SL-structure, where SL is a complete homogeneous Lorentz group. If M is (Minkowski) space-time, so a smooth oriented (compact) manifold of hyperbolic type, $SL(p,q)(SO(p,q)) \to F(M) \xrightarrow{\pi} M$ is a principal bundle of orthonormal frames. *Spin structure* exists if there is an extension

$$\begin{array}{ccccc} \mathrm{Spin}(p,q) & \to & \hat{F}(M) & \to & M \\ \downarrow & & \downarrow & & \downarrow \\ SL(p,q) & \to & F(M) & \to & M \\ SO(p,q) & & & & \end{array}$$

where $\hat{F}(M)$ is the double covering of $F(M)$ and $\mathrm{Spin}(p,q)$ is a non-trivial double covering of $SL(p,q)(SO(p,q))$—i.e.

$$1 \to \mathbf{Z}_2(=\pi_1\, SL(p,q)(SO(p,q))) \xrightarrow{i} \mathrm{Spin}(p,q) \xrightarrow{\sigma} SL(p,q)(SO(p,q)) \to 1$$

$(p+q \geqslant 3)$; that is, when an $SL(SO)$-bundle can be lifted to a Spin-bundle. The set of isomorphism classes of double coverings, e.g. spin structures, is in one-one correspondence with elements of the cohomology set $H^1(F(M), \mathbf{Z}_2)$, which have a non-trivial restriction to $H^1(SO(n), \mathbf{Z}_2)$. From the exact sequence [see Milnor (1963), Bichteler (1968), Geroch (1968) and Atiyah-Bott (1968)]

$$\cdots \to H^1(M, \mathbf{Z}_2) \xrightarrow{\pi^*} H^1(F(M), \mathbf{Z}_2) \xrightarrow{i^*} H^1(SL(n)(SO(n)), \mathbf{Z}_2) \xrightarrow{\tau} H^2(M, \mathbf{Z}_2) \to \cdots$$

where τ is the transgression in fibering $F(M) \to M$, or from the exact sequence [see, for example, Lichnerowicz (1968) and Kamber & Tondeur (1967)]

$$\cdots \to H^1(M, \pi_1\, SL(n)(SO(n)) = \mathbf{Z}_2) \to H^1(M, \mathrm{Spin}(n)) \xrightarrow{\sigma^*} \mathrm{H}^1(M, \underline{\mathrm{SL}}(n)(\underline{\mathrm{SO}}(n))) \xrightarrow{\delta} H^2(M, \pi_1\, SL(n)(SO(n)) = \mathbf{Z}_2) \to \cdots$$

(where $\underset{\sim}{G}$ denotes the G-valued sheaf of germs of continuous functions $M \to G$), we have $\delta(\xi) = -\tau_\xi(\iota)$ is the obstruction of the lifting of an $SL(SO)$-bundle ξ on M to a Spin-bundle [where M is a CW-complex and ι is the fundamental class of $SO(n)$ with natural orientation—$\iota \in H^1(SO(n), \pi_1 SO(n)) \simeq \mathrm{Hom}(\pi_1 SO(n), \pi_1 SO(n))$]. That is, ξ can be lifted to a Spin(n)-bundle iff $\delta(\xi) = w_2(\xi) \in H^2(M, \mathbf{Z}_2)$, the second Stiefel-Whitney class, vanishes. Thus we have

Proposition 7 (Borel-Hirzebruch, Haefligher, Shapiro, and others). M has spin structure iff $\delta = w_2(M) = 0$.

Since $\ker i^* = \pi^* H^1(M, \mathbf{Z}_2)$, we have

Proposition 8 (Milnor). If M is a spin manifold, then the number of distinct spin structures is equal to the number of elements of $H^1(M, \mathbf{Z}_2)$.

Let $V_i \in H^i(M^n, \mathbf{Z}_2)$ be the Wu class of $u \in H^{n-i}(M^n, \mathbf{Z}_2)$ [i.e. $u \cup V_i$ (cup product) $= Sq^i(u)$, where Sq^i is Steenrod operator; cf. Spanier (1966)]; so, $w_{2q}(M) = Sq^q V_q = V_q^2$; etc. As M^{2n} is symplectic iff for every class $u \in H^n(M^{2n}, \mathbf{Z}_2)$, $u^2 = 0$, which occurs iff $V_n = 0$, we have

Lemma 9. M^{2n} is symplectic iff $V_n = 0$.

We recall the result of Massey (1960), which states that if M^n is a compact, orientable smooth manifold *sans* boundary, then $w_{n-1}(M) = 0$ for n even and $n \equiv 2 \bmod 4$. Thus if M^n is spin manifold, then $Sq^2 H^{n-2}(M, \mathbf{Z}_2) = 0$, so $V_{4k+2} = 0$ for $k \geqslant 0$. Therefore, for $2n \equiv 4 \bmod 8$ we have

Proposition 10. If M^{8k+4} has spin structure, then M^{8k+4} is symplectic.

Compare Thomas (1967); for further details see Hurt (1969a).

Thus a class of symplectic (Hamiltonian) dynamical systems is given by manifolds M^{8k+4} which carry spin structure.

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